

A CONSTRUCTION BY DEFORMATION OF UNITARY IRREDUCIBLE REPRESENTATIONS OF $SU(n+1)$

BENJAMIN CAHEN

To the memory of my father, Alfred Cahen (1932-2016)

ABSTRACT. We derive unitary irreducible representations of $SU(n+1)$ from a minimal realization of $sl(n+1, \mathbb{C})$ by using various techniques from deformation theory.

1. INTRODUCTION

The deformations of Lie algebras were intensively studied in the years 1960-70 [10], [21], [22], [18] and still remain objects of active research, see for instance [7], [8] and [6]. On the other hand, the deformations of Lie algebra representations have not been studied as systematically, with some notable exceptions, see [23], [13], [19] and also [20].

Constructing (formal) deformations of Lie algebra representations is a way to derive a family of representations from a given one and then to get many representations from a few ones. However, the existence and classification problems for deformations depend on some Lie algebra cohomology modules which are not easy to compute in general, see for instance [19] and [3].

The aim of the present note is to use deformation theory in order to recover some unitary irreducible representations of $SU(n+1)$. More precisely, we construct the unitary irreducible representations of $SU(n+1)$ considered in [4] by deformation of a so-called minimal realization of $sl(n+1, \mathbb{C})$ [15]. It is known that such a minimal realization is connected to the minimal (non trivial) nilpotent coadjoint orbit of $SL(n+1, \mathbb{C})$ [16], [1], [17]. By taking a parametrization of this orbit, we can simplify the computation of the deformations of the minimal realization by using the Moyal star product and the Weyl correspondence as in [1] and [3].

Although the derivation presented here can be considered as a simple exercise in deformation theory, we have not found it explicitly done in details in the literature. Moreover, we could hope for applications of this method to the description of representations of Lie algebras (in particular of unitary dual of Lie groups) in more general situations (some examples can already be found in [20] and [3]).

This note is organized as follows. In Section 2, we describe the unitary irreducible representations of $SU(n+1)$ introduced in [4] as an analogue to the holomorphic discrete series representations of $SU(1, n)$ and we compute their differentials which can be extended to representations of $sl(n+1, \mathbb{C})$. In the notation of [11], p. 143, these representations

2000 *Mathematics Subject Classification.* 17B10; 17B20; 17B56; 22E46; 53D55.

Key words and phrases. Deformation of representation; Lie algebra; unitary group; Chevalley-Eilenberg cohomology; Moyal star product; Weyl correspondence; minimal realization; minimal coadjoint orbit.

have highest weight $m\epsilon_1$ with m integer ≥ 1 . Section 3 is devoted to some generalities on (formal) deformations of Lie algebra homomorphisms and, in Section 4, we recall the Moyal star product and the Weyl correspondence [9], [24]. In Section 5, we show how a symplectic chart of the minimal nilpotent coadjoint orbit of $sl(n+1, \mathbb{C})$ naturally leads to a minimal realization of $sl(n+1, \mathbb{C})$ and we expose the method which will be used to recover the representations of $SU(n+1)$. In Section 6, we compute the first cohomology module corresponding to the deformation of the minimal realization and then we derive the desired representations of $SU(n+1)$ in Section 7. In particular, by this way we can recover all the irreducible unitary representations of $SU(2)$.

2. REPRESENTATIONS OF $SU(n+1)$

Here we consider a family of representations of $SU(n+1)$ indexed by an integer $m \geq 1$. In [4], we showed that this family can be contracted to the unitary irreducible representations of the Heisenberg group of dimension $2n+1$ as the holomorphic discrete series representations of $SU(1, n)$.

The group $SU(n+1)$ consists of all complex $(n+1) \times (n+1)$ matrices g with determinant 1 such that $g^*g = I_{n+1}$. Here we write the elements of the group $SU(n+1)$ as block matrices

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with matrices $a(1 \times 1)$, $b(1 \times n)$, $c(n \times 1)$ and $d(n \times n)$.

The Lie algebra $\mathfrak{su}(n+1)$ of $SU(n+1)$ consists of all matrices of the form

$$\begin{pmatrix} i\alpha & b \\ -b^* & A \end{pmatrix}$$

where $\alpha \in \mathbb{R}$, $b \in \mathbb{C}^n$ and A is an anti-Hermitian $n \times n$ matrix (that is, $A^* = -A$) such that $i\alpha + \text{Tr}(A) = 0$.

The group $SU(n+1)$ acts naturally on the projective space $\mathbb{P}_n(\mathbb{C})$ and this action induces an holomorphic action (defined almost everywhere) of $SU(n+1)$ on \mathbb{C}^n by fractional linear transformations

$$g \cdot z = (a + bz^t)^{-1}(c + dz^t)^t, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where the subscript t denotes transposition.

For each integer $m \geq 1$, let \mathcal{P}_m be the space of all complex polynomial functions on \mathbb{C}^n of degree $\leq m$. We endow \mathcal{P}_m with the Hilbert product

$$\langle f_1, f_2 \rangle_m := \int_{\mathbb{C}^n} f_1(z) \overline{f_2(z)} d\mu_m(z)$$

where the measure μ_m on \mathbb{C}^n is defined by

$$d\mu_m(z) := \frac{(m+1) \dots (m+n)}{\pi^n} (1 + \|z\|^2)^{-m-n-1} dx_1 dy_1 \dots dx_n dy_n.$$

Here we use the notation $z = (x_1 + iy_1, x_2 + iy_2, \dots, x_n + iy_n)$ where $x_k, y_k \in \mathbb{R}$ for $k = 1, 2, \dots, n$.

Now, let π_m be the representation of $SU(n+1)$ on \mathcal{P}_m defined by

$$(\pi_m(g)f)(z) = (bz^t + a)^m f(g^{-1} \cdot z), \quad g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We can easily verify that π_m is unitary. Moreover, the differential $d\pi_m$ of π_m can be extended to a representation of $sl(n+1, \mathbb{C}) = su(n+1)^c$ also denoted by $d\pi_m$. We have

$$(d\pi_m(X)f)(z) = -m(\beta z^t + \alpha)f(z) + df(z)((\alpha + \beta z^t)z - (\gamma + \delta z^t)^t)$$

where

$$X = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

with matrices $\alpha(1 \times 1)$, $\beta(1 \times n)$, $\gamma(n \times 1)$ and $\delta(n \times n)$.

In order to give more explicit formulas for $d\pi_m$, let us introduce the following basis of $sl(n+1, \mathbb{C})$. For $1 \leq i, j \leq n+1$, write E_{ij} for the matrix whose ij -th entry is 1 and all of the other entries are 0. Then the matrices $H_k = E_{k+1k+1} - E_{11}$ ($1 \leq k \leq n$) form a basis for the Cartan subalgebra \mathfrak{h} of $sl(n+1, \mathbb{C})$ consisting of all diagonal matrices of $sl(n+1, \mathbb{C})$ and, obviously, the matrices H_k ($1 \leq i \leq n$) and E_{ij} ($1 \leq i \neq j \leq n+1$) form a basis for $sl(n+1, \mathbb{C})$. Then we have

$$\begin{aligned} (d\pi_m(H_k)f)(z) &= mf(z) - z_k \frac{\partial f}{\partial z_k} - \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j} \\ (d\pi_m(E_{1k+1})f)(z) &= -mz_k f(z) + z_k \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j} \\ (d\pi_m(E_{k+11})f)(z) &= -\frac{\partial f}{\partial z_k} \\ (d\pi_m(E_{i+1j+1})f)(z) &= -z_j \frac{\partial f}{\partial z_i} \end{aligned}$$

for $1 \leq k \leq n$ and $1 \leq i \neq j \leq n$.

We can easily see that $d\pi_m$ -hence π_m - is irreducible. Indeed, let \mathcal{V} be a nonzero subspace of \mathcal{P}_m which is invariant under $d\pi_m(X)$ for each $X \in sl(n+1, \mathbb{C})$. Then there exists at least one nonzero element f in \mathcal{V} . Thus, by applying the operators $d\pi_m(E_{k+11})$ to f , we get $1 \in \mathcal{V}$ and by applying the operators $d\pi_m(E_{1k+1})$ and $d\pi_m(E_{i+1j+1})$ to 1 we see that $\mathcal{V} = \mathcal{P}_m$.

Let us denote by ϵ_k , $1 \leq k \leq n$, the linear form on \mathfrak{h} defined by

$$\epsilon_k : \text{Diag}(a_1, a_2, \dots, a_{n+1}) \rightarrow a_k.$$

It is well-known that the root system of $sl(n+1, \mathbb{C})$ relative to \mathfrak{h} is

$$\Delta = \{\epsilon_i - \epsilon_j : 1 \leq i, j \leq n+1\},$$

see for instance [11]. The ordering on Δ is usually taken so that the positive roots are $\epsilon_i - \epsilon_j$ ($1 \leq i < j \leq n+1$). In this context, we can verify that $d\pi_m$ has highest weight $m\epsilon_1$ and highest weight vector $f = z_n^m$.

3. GENERALITIES ON DEFORMATIONS

In this section, we recall some definitions and results of deformation theory. The material of this section is essentially taken from [23], [13], [19], see also [12] and [3].

Let \mathfrak{g} be a Lie algebra over \mathbb{C} and let A be an associative algebra over \mathbb{C} with unit element 1. Then A is also a Lie algebra for the commutator $[a, b] := ab - ba$. Let $\varphi : \mathfrak{g} \rightarrow A$ be a Lie algebra homomorphism.

Definition 3.1. (1) A formal deformation of φ is a formal series $\Phi = \sum_{k \geq 0} t^k \Phi_k$ where $\Phi_0 = \varphi$ and Φ_k is a linear map from \mathfrak{g} to A for each $k \geq 1$, such that

$$(3.1) \quad \Phi([X, Y]) = [\Phi(X), \Phi(Y)]$$

for each X and Y in \mathfrak{g} . Here we have extended the bracket of A to formal series by bilinearity.

(2) Two formal deformations Φ and Ψ of φ are said to be equivalent if there exists a series $a = 1 + ta_1 + t^2a_2 + \dots \in A[[t]]$ such that for each $X \in \mathfrak{g}$, we have

$$(3.2) \quad a^{-1}\Phi(X)a = \Psi(X).$$

The study of the formal deformations of φ naturally leads us to consider the structure of \mathfrak{g} -module on A defined by $X \cdot a = [\varphi(X), a]$ for $X \in \mathfrak{g}$ and $a \in A$ and the Chevalley-Eilenberg cohomology of \mathfrak{g} with values in the \mathfrak{g} -module A . Indeed, denoting by ∂ the corresponding cobord operator, we immediately see that Eq. 3.1 is equivalent to the fact that for each $n \geq 0$ and each $X, Y \in \mathfrak{g}$, we have

$$\begin{aligned} (\partial\Phi_n)[X, Y] &:= [\varphi(X), \Phi_n(Y)] + [\Phi_n(X), \varphi(Y)] - \Phi_n([X, Y]) \\ &= - \sum_{k=1}^{n-1} [\Phi_k(X), \Phi_{n-k}(Y)]. \end{aligned}$$

In particular, we see that if such a deformation Φ exists then Φ_1 is a 1-cocycle.

We have the following result, see for instance [13], Section III and [19], Section I.

Proposition 3.2. (1) If we have $H^2(\mathfrak{g}, A) = (0)$ then, for each 1-cocycle $\alpha : \mathfrak{g} \rightarrow A$, there exists a formal deformation Φ such that $\Phi_1 = \alpha$.

(2) If we have $H^1(\mathfrak{g}, A) = (0)$ then each formal deformation Φ of φ is equivalent to φ .

In [3], we proved the following result.

Proposition 3.3. Assume that $H^1(\mathfrak{g}, A)$ is one-dimensional and that there exists a formal deformation Φ of φ such the class of Φ_1 generates $H^1(\mathfrak{g}, A)$. For each sequence $c = (c_k)_{k \geq 1}$ of complex numbers, consider the formal series $S_c(t) := \sum_{k \geq 1} c_k t^k$ and the formal deformation Φ^c of φ defined by $\Phi^c(X) = \sum_{r \geq 0} S_c(t)^r \Phi_r(X)$ for each $X \in \mathfrak{g}$.

Then the map $c \rightarrow \Phi^c$ is a bijection from the set of all sequences $c = (c_k)_{k \geq 1}$ of \mathbb{C} onto the set of all equivalence classes of formal deformations of φ .

Note that the preceding definitions and results can be applied to the particular case of a representation φ of \mathfrak{g} in a complex vector space V , since φ is also a Lie algebra homomorphism from \mathfrak{g} to $\text{End}(V)$, or, more generally, to a subalgebra A of $\text{End}(V)$.

4. WEYL CORRESPONDENCE AND MOYAL STAR PRODUCT

Here we first recall the Moyal star product, see for instance [2]. Take coordinates (p, q) on $\mathbb{R}^{2n} \cong \mathbb{R}^n \times \mathbb{R}^n$ and let $x = (p, q)$. Then one has $x_i = p_i$ for $1 \leq i \leq n$ and $x_{i+n} = q_{i-n}$

for $n+1 \leq i \leq 2n$. For $u, v \in C^\infty(\mathbb{R}^{2n})$, define $P^0(u, v) := uv$,

$$P^1(u, v) := \sum_{k=1}^n \left(\frac{\partial u}{\partial p_k} \frac{\partial v}{\partial q_k} - \frac{\partial u}{\partial q_k} \frac{\partial v}{\partial p_k} \right) = \sum_{1 \leq i, j \leq n} \Lambda^{ij} \partial_{x_i} u \partial_{x_j} v$$

(the Poisson brackets) and, more generally, for $l \geq 2$,

$$P^l(u, v) := \sum_{1 \leq i_1, \dots, i_l, j_1, \dots, j_l \leq n} \Lambda^{i_1 j_1} \Lambda^{i_2 j_2} \dots \Lambda^{i_l j_l} \partial_{x_{i_1} \dots x_{i_l}}^l u \partial_{x_{j_1} \dots x_{j_l}}^l v.$$

Then the Moyal product $*_M$ is the following formal deformation of the pointwise multiplication of $C^\infty(\mathbb{R}^{2n})$

$$u *_M v := \sum_{l \geq 0} \frac{t^l}{l!} P^l(u, v)$$

where t is a formal parameter. Moreover, the corresponding Moyal brackets are given by

$$[u, v]_{*_M} := \frac{1}{2t} (u *_M v - v *_M u) = \sum_{l \geq 0} \frac{t^{2l}}{(2l+1)!} P^{2l+1}(u, v).$$

Now, we restrict $*_M$ to polynomials on \mathbb{R}^{2n} and take $t = -i/2$. Then we get an associative product $*$ on polynomials which we denote by $*$. This product corresponds to the composition of operators in the usual Weyl quantization procedure as we will explain below.

The Weyl correspondence on \mathbb{R}^{2n} is defined as follows, see [5], [9], [14]. For each f in the Schwartz space $\mathcal{S}(\mathbb{R}^{2n})$, we define the operator $W(f)$ acting on the Hilbert space $L^2(\mathbb{R}^n)$ by

$$W(f)\varphi(p) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{isq} f(p + (1/2)s, q) \varphi(p + s) ds dq.$$

As it is well-known, that the Weyl calculus can be extended to much larger classes of symbols (see for instance [14]). In particular, if $f(p, q) = u(p)q^\alpha$ where $u \in C^\infty(\mathbb{R}^n)$ then we have

$$(4.1) \quad W(f)\varphi(p) = \left(i \frac{\partial}{\partial s} \right)^\alpha (u(p + (1/2)s) \varphi(p + s)) \Big|_{s=0},$$

see [24]. For instance, if $f(p, q) = u(p)$ then $W(f)\varphi(p) = u(p) \varphi(p)$ and if $f(p, q) = u(p)q_k$ then

$$(4.2) \quad W(f)\varphi(p) = i((1/2)\partial_k u(p) \varphi(p) + u(p)\partial_k \varphi(p)).$$

Moreover, we have $W(f_1 * f_2) = W(f_1)W(f_2)$ for each functions f_1, f_2 on \mathbb{R}^{2n} of the form $u(p)q^\alpha$, in particular for polynomials, see [9], p. 103.

Note also that, since the map W and the product $*$ on polynomials can be defined in a purely algebraic way, see Eq. 4.1, we can extended them to the polynomials in complex variables p, q without any modification.

5. MINIMAL REALIZATION

In [1], a general method for constructing minimal realizations of semisimple complex Lie algebras from minimal coadjoint orbits was introduced. In the particular case of the Lie algebra $\mathfrak{g} := \mathfrak{sl}(n+1, \mathbb{C})$ of $G := SL(n+1, \mathbb{C})$, this method goes as follows.

First, we can identify the dual \mathfrak{g}^* of \mathfrak{g} with \mathfrak{g} by means of the bilinear form on \mathfrak{g} defined by $\langle X, Y \rangle := \text{Tr}(XY)$. In this identification, the coadjoint action of G corresponds to the adjoint action of G and the coadjoint orbits to the adjoint orbits.

This is a simple exercise to show that the minimal (non trivial) nilpotent (co)adjoint orbit \mathcal{O} of G consists of all rank one matrices of \mathfrak{g} .

Now, let us consider the map Ψ from \mathbb{C}^{2n} to $\mathcal{O}' := \mathcal{O} \cup (0)$ defined by

$$\Psi(p, q) := \begin{pmatrix} -\sum_{j=1}^n p_j q_j & q_1 & \cdots & q_n \\ -p_1 \sum_{j=1}^n p_j q_j & p_1 q_1 & \cdots & p_1 q_n \\ \vdots & \vdots & \ddots & \vdots \\ -p_n \sum_{j=1}^n p_j q_j & p_n q_1 & \cdots & p_n q_n \end{pmatrix}.$$

Then the image of Ψ is a dense open subset of \mathcal{O}' .

For each $X \in \mathfrak{g}$, let us denote by \tilde{X} the corresponding coordinate function on \mathbb{C}^{2n} :

$$\tilde{X}(p, q) := \langle \Psi(p, q), X \rangle.$$

Proposition 5.1. (1) *For each $X, Y \in \mathfrak{g}$, we have*

$$[\tilde{X}, \tilde{Y}]_* = \{\tilde{X}, \tilde{Y}\} = [X, Y].$$

(2) *The map $\rho_0 : X \rightarrow W(i\tilde{X})$ is a representation of \mathfrak{g} in $\mathbb{C}[q] := \mathbb{C}[q_1, q_2, \dots, q_n]$.*

Proof. (1) Let X and Y in \mathfrak{g} . The equation $\{\tilde{X}, \tilde{Y}\} = [X, Y]$ can be verified by a direct computation. On the other hand, since \tilde{X} and \tilde{Y} are polynomials of degree ≤ 1 in the variables q_1, q_2, \dots, q_n , we have $P^k(\tilde{X}, \tilde{Y}) = 0$ for each $k \geq 3$, hence we get $[\tilde{X}, \tilde{Y}]_* = \{\tilde{X}, \tilde{Y}\}$.

(2) Let X and Y in \mathfrak{g} . By (1), we have

$$(i\tilde{X}) * (i\tilde{Y}) - (i\tilde{Y}) * (i\tilde{X}) = i[X, Y].$$

Then, by the remark at the end of Section 4, we get $[W(i\tilde{X}), W(i\tilde{Y})] = W(i[X, Y])$ hence the result. \square

The representation ρ_0 is a minimal realization of \mathfrak{g} , that is, a realization of \mathfrak{g} as Lie algebra of differential operators acting on functions of n variables with n minimal, see [15].

Note that

$$\text{Span}\{E_{n+12}, \dots, E_{n+1n}, E_{21}, \dots, E_{n+11}\}$$

is a Heisenberg Lie algebra of dimension $2n - 1$ with central element E_{n+11} , the only non trivial brackets being $[E_{n+1k}, E_{k1}] = E_{n+11}$ for $k = 2, \dots, n$. Then Ψ was chosen so that $\tilde{E}_{n+1k} = p_{k-1}q_n$, $\tilde{E}_{k1} = q_{k-1}$ (for $k = 2, \dots, n$) and $\tilde{E}_{n+11} = q_n$. In fact, these conditions determine Ψ uniquely.

Now, we aim to study the deformations of ρ_0 . By using the map $f \rightarrow W(if)$ this is equivalent to studying the deformations of the Lie algebra homomorphism $X \rightarrow \Phi_0(X) := \tilde{X}$ from \mathfrak{g} to $M := \mathbb{C}[p, q]$ endowed with $[\cdot, \cdot]_*$.

As explained in Section 3, we endow M with the \mathfrak{g} -module structure defined by $X \cdot f := [\tilde{X}, f]_*$ and then consider the corresponding Chevalley-Eilenberg cohomology.

6. DETERMINATION OF $H^1(\mathfrak{g}, M)$

Recall that $H^1(\mathfrak{g}, M)$ is the quotient space $Z^1(\mathfrak{g}, M)/B^1(\mathfrak{g}, M)$ where $Z^1(\mathfrak{g}, M)$ consists of all linear maps $\varphi : \mathfrak{g} \rightarrow M$ satisfying

$$(6.1) \quad \partial\varphi(X, Y) := [\tilde{X}, \varphi(Y)]_* + [\varphi(X), \tilde{Y}]_* - \varphi[X, Y] = 0$$

(the 1-cocycles) and $B^1(\mathfrak{g}, M)$ consists of all maps from \mathfrak{g} to M of the form $X \rightarrow [\tilde{X}, f]_*$ for $f \in M$ (the 1-coboundaries).

The aim of this section is to compute $H^1(\mathfrak{g}, M)$. We begin with the following 'Poincaré lemma'.

Lemma 6.1. *Let $F_i(q)$, $i = 1, 2, \dots, n$ be a family of polynomials in the variable $q = (q_1, q_2, \dots, q_n)$ such that, for each $i, j = 1, 2, \dots, n$, one has $\frac{\partial F_i}{\partial q_j} = \frac{\partial F_j}{\partial q_i}$. Then there exists a polynomial $F(q)$ such that $\frac{\partial F}{\partial q_i} = F_i$ for each $i = 1, 2, \dots, n$.*

Proof. By the usual Poincaré lemma, the result is true for polynomials in real variables q_i which implies that it is also true for polynomials in complex variables q_i . \square

Proposition 6.2. *The space $H^1(\mathfrak{g}, M)$ is one dimensional, generated by the class of the cocycle φ_1 defined by $\varphi_1(E_{11} - E_{22}) = 1$, $\varphi_1(E_{kk} - E_{k+1k+1}) = 0$ for $k = 2, \dots, n$, $\varphi_1(E_{1k+1}) = p_k$ for $k = 1, 2, \dots, n$ and $\varphi_1(E_{ij}) = 0$ for $i \geq 2$.*

Proof. We have divided the proof into several steps. The method of the proof is quite elementary and consists in transforming progressively a given 1-cocycle to an equivalent one which is more simple by adding suitable 1-coboundaries.

Let us consider a 1-cocycle $\varphi : \mathfrak{g} \rightarrow M$.

1) First we apply Eq. 6.1 to $X = E_{k+11}$ and $Y = E_{l+11}$ for $k, l = 1, 2, \dots, n$. Writing $\varphi_k = \varphi(E_{k+11})$ for simplicity, we get $\frac{\partial \varphi_k}{\partial p_l} = \frac{\partial \varphi_l}{\partial p_k}$ for each $k, l = 1, 2, \dots, n$. Then, by decomposing each φ_k as $\varphi_k = \sum_{\alpha} \varphi_k^{\alpha}(p) q^{\alpha}$ with the usual multi-index notation, we have $\frac{\partial \varphi_k^{\alpha}}{\partial p_l} = \frac{\partial \varphi_l^{\alpha}}{\partial p_k}$ for each k, l, α .

Thus, by Lemma 6.1, for each α there exists a polynomial $\varphi^{\alpha}(p)$ such that $\frac{\partial \varphi_k^{\alpha}}{\partial p_k} = \varphi_k^{\alpha}$ for each $k = 1, 2, \dots, n$.

Now, let $\phi := \sum_{\alpha} \varphi^{\alpha}(p) q^{\alpha}$. For each $k = 1, 2, \dots, n$, we have

$$[\phi, q_k]_* = \frac{\partial \phi}{\partial p_k} = \varphi_k.$$

Hence, replacing φ by the equivalent 1-cocycle $\varphi - [\phi, \cdot]_*$, we can always assume that $\varphi(E_{k+11}) = 0$ for each $k = 1, 2, \dots, n$.

2) We apply Eq. 6.1 to $X = E_{kl}$, $k, l \geq 2$, $k \neq l$ and $Y = E_{j+11}$, $j = 1, 2, \dots, n$. Taking 1) into account, we can immediately see that $\varphi(E_{kl})$ is a polynomial in the variables q_1, q_2, \dots, q_n .

3) Similarly, applying Eq. 6.1 to $X \in \mathfrak{h}$ and $Y = E_{j+11}$, we verify that $\varphi(X)$ is a polynomial in the variables q_1, q_2, \dots, q_n .

4) Now, we fix $k = 1, 2, \dots, n-1$ and we apply Eq. 6.1 to $X = E_{n+1k+1}$ and $Y = \sum_{j=1}^{n-1} (E_{n+1n+1} - E_{j+1j+1})$. Write $\varphi_k := \varphi(E_{n+1k+1})$ for simplicity and recall that φ_k is

a polynomial in q_1, q_2, \dots, q_n by 2). Then we see that there exists a polynomial $u_k(q)$ in q_1, q_2, \dots, q_n such that

$$(6.2) \quad -n\varphi_k = \sum_{j=1}^n q_j \frac{\partial \varphi_k}{\partial q_j} + q_n u_k(q).$$

Let $\varphi_k = \sum_m \varphi_k^m$ and $u_k = \sum_m u_k^m$ be the decompositions of φ_k and u_k into homogeneous polynomials of degree m in q_1, q_2, \dots, q_n . Then Eq. 6.2 implies that

$$-n \sum_m \varphi_k^m = \sum_m m \varphi_k^m + q_n u_{k-1}(q)$$

and we conclude that, for each $k = 1, 2, \dots, n-1$, there exists a polynomial ψ_k in q_1, q_2, \dots, q_n such that $\varphi_k = q_n \psi_k$.

Taking $X = E_{n+1k+1}$ and $Y = E_{n+1l+1}$ in Eq. 6.1 for $k, l = 1, 2, \dots, n-1$, we get $\frac{\partial \psi_k}{\partial q_l} = \frac{\partial \psi_l}{\partial q_k}$ for each k, l . This implies the existence of a polynomial ψ in q_1, q_2, \dots, q_n such that $\psi_k = \frac{\partial \psi}{\partial q_k}$ for each $k = 1, 2, \dots, n-1$.

Thus, by replacing φ by $\varphi - [\cdot, \psi]_*$, we are led to the case where $\varphi(E_{n+1k+1}) = 0$ for each $k = 1, 2, \dots, n-1$ and the condition $\varphi(E_{k+11}) = 0$ for each $k = 1, 2, \dots, n$ is still satisfied.

5) Let $k = 1, 2, \dots, n, l = 1, 2, \dots, n-1$ with $k \neq l$. By applying Eq. 6.1 to $X = E_{k+1l+1}$ and $Y = E_{n+1j+1}$ for $j = 1, 2, \dots, n-1$, we see that $\varphi(E_{k+1l+1})$ only depends on q_n . Thus, taking into account the equality

$$[E_{k+1l+1}, E_{l+1k+1}] = E_{k+1k+1} - E_{l+1l+1}$$

we get $\varphi(E_{k+1k+1} - E_{l+1l+1}) = 0$. Hence, applying Eq. 6.1 to $X = E_{k+1k+1} - E_{l+1l+1}$ and $Y = E_{k+1l+1}$ we obtain $\varphi(E_{k+1l+1}) = 0$.

Finally, we apply Eq. 6.1 to $X = E_{j+1n+1}$ and $Y = E_{k+1j+1}$ and we also obtain $\varphi(E_{k+1n+1}) = 0$.

6) Now, take $X \in \mathfrak{h}$ and $Y = E_{k+1j+1}$ in Eq. 6.1. Then we see that $\varphi(X)$ only depends on q_n .

Let $H_0 = E_{11} - E_{22} \in \mathfrak{h}$. Then we can replace φ by $\varphi + [\cdot, F(q_n)]_*$ for a suitable polynomial $F(q_n)$ so that $\varphi(H_0)$ is a constant which we denote by a .

7) Taking $X = E_{12}$ and successively $Y = E_{k+11}$, ($k = 1, 2, \dots, n$) and $Y = E_{n+1k}$, ($k = 2, \dots, n-1$) in Eq. 6.1, we see that

$$\varphi(E_{12}) = ap_1 + f(q_n)$$

where $f(q_n)$ is a polynomial. Moreover, taking also $X = E_{12}$ and $Y = H_0$, we get $-2f(q_n) = q_n \frac{\partial f}{\partial q_n}$ hence $f = 0$ and $\varphi(E_{12}) = ap_1$.

8) Finally, we apply Eq. 6.1 to $X = E_{12}$ and $Y = E_{2k+1}$ where $k = 2, \dots, n$ we obtain $\varphi(E_{1k+1}) = ap_k$.

□

7. DERIVATION OF THE REPRESENTATIONS π_m

In this section, we retain the notation of the previous sections. Proposition 6.2 leads us to consider the formal deformations Φ of $\Phi_0 : X \rightarrow \tilde{X}$ such that $\Phi_1 = a\varphi_1$ for $a \in \mathbb{C}$. We have the following result.

Proposition 7.1. *For each $a \in \mathbb{C}$, the map $\Phi_a : \mathfrak{g} \rightarrow M[[t]]$ defined by $\Phi_a(X) = \tilde{X} + ta\varphi_1(X)$ is a formal deformation of Φ_0 in M .*

Proof. Taking into account that φ_1 is a 1-cocycle (see Section 6), the result follows immediately from the equality $[\varphi_1(X), \varphi_1(Y)]_* = 0$ for $X, Y \in \mathfrak{g}$. \square

By using the properties of W (see Section 4), we get the following proposition.

Proposition 7.2. *For each $a \in \mathbb{C}$, let $m(a) := -1/2(a+n+1)$. Then the map ρ_a defined by*

$$\rho_a(X) = W \left(\tilde{X} - \frac{i}{2}a\varphi_1(X) \right)$$

for $X \in \mathfrak{g}$ is a representation of \mathfrak{g} in $\mathbb{C}[p]$ and we have

$$\begin{aligned} (\rho_a(H_k)f)(z) &= m(a)f(z) - z_k \frac{\partial f}{\partial z_k} - \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j} \\ (\rho_m(E_{1k+1})f)(z) &= -m(a)z_k f(z) + z_k \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j} \\ (\rho_a(E_{k+11})f)(z) &= -\frac{\partial f}{\partial z_k} \\ (\rho_a(E_{i+1j+1})f)(z) &= -z_j \frac{\partial f}{\partial z_i} \end{aligned}$$

for $1 \leq k \leq n$ and $1 \leq i \neq j \leq n$.

Proof. The fact that ρ_a is a representation \mathfrak{g} follows from Proposition 7.2 and the formulas for ρ_a can be easily verified by Eq. 4.2. \square

In other words, the formulas for ρ_a are the same as the formulas for $d\pi_m$, see Section 2, but note that these two representations don't act on the same spaces since ρ_a acts on $\mathbb{C}[p]$ and $d\pi_m$ on the finite dimensional space \mathcal{P}_m .

In order to recover the representations π_m of Section 2, we select now the values of a (or, equivalently, of $m(a)$) for which there exists a non trivial finite dimensional subspace of $\mathbb{C}[p]$ that is invariant under ρ_a .

Proposition 7.3. *Let $a \in \mathbb{C}$. Assume that \mathcal{P} is a non trivial finite dimensional subspace of $\mathbb{C}[p]$ that is invariant under ρ_a . Then $m(a)$ is a non negative integer, we have $\mathcal{P} = \mathcal{P}_{m(a)}$ and the restriction of ρ_a to \mathcal{P} coincides with $d\pi_{m(a)}$.*

Proof. Let $a \in \mathbb{C}$. Let $\mathcal{P} \neq (0)$ a finite dimensional subspace of $\mathbb{C}[p]$ which is invariant under ρ_a . Define $m := \max\{\deg(f) : f \in \mathcal{P} \setminus (0)\}$. Let f be an element of \mathcal{P} of degree m . Let us decompose f as $f = \sum_{k=0}^m f_k$ where, for each k , f_k is an homogeneous polynomial of degree k . Then we have $f_m \neq 0$ and

$$\rho_a(E_{1l+1})f = p_l \sum_{k=0}^m (k - m(a))f_k.$$

We see that if $m \neq m(a)$, we get a contradiction. Thus we have $m(a) = m$ hence $m(a)$ is a non negative integer and $\mathcal{P} \subset \mathcal{P}_{m(a)}$. Since $\mathcal{P}_{m(a)}$ is irreducible under the action of $d\pi_{m(a)}$, see Section 2, we can conclude that $\mathcal{P} = \mathcal{P}_{m(a)}$. \square

Then we have recovered the representations $d\pi_m$ of Section 2, hence the representations π_m by integration. Note that by taking $n = 1$ we see that this method gives all the unitary irreducible representations of $SU(2)$.

REFERENCES

- [1] D. Arnal, H. Benamor and B. Cahen, Minimal realizations of classical simple Lie algebras through deformations, *Ann. Fac. Sci. Toulouse VII*, 2 (1998), 169-184.
- [2] D. Arnal and J.-C. Cortet, Représentations * des groupes de Lie exponentiels, *J. Funct. Anal.* 92, 1 (1990), 103-135.
- [3] B. Cahen, Déformations formelles de certaines représentations de l'algèbre de Lie d'un groupe de Poincaré généralisé, *Ann. Math. Blaise Pascal* 8, 1 (2001), 17-37.
- [4] B. Cahen, Contractions of $SU(1, n)$ and $SU(n+1)$ via Berezin quantization, *J. Anal. Math.* 97 (2005) 83-102.
- [5] M. Combesure and D. Robert, *Coherent States and Applications in Mathematical Physics*, Springer, 2012.
- [6] D. Burde, Contractions of Lie algebras and algebraic groups, *Arch. Math.*, Brno 43, 5 (2007), 321-332.
- [7] A. Fialowski, Deformations in Mathematics and Physics, *Intern. Journ. Theor. Physics*, 47, 2 (2008), 333-337.
- [8] A. Fialowski and M. Penkava, Deformations of nilpotent associative algebras of dimension 4, *Linear Algebra Appl.* 457 (2014), 408-427.
- [9] B. Folland, *Harmonic Analysis in Phase Space*, Princeton Univ. Press, 1989.
- [10] M. Gerstenhaber, On the deformation of rings and algebras, *Ann. Math.* 79, 1 (1964), 59-103.
- [11] R. Goodman and N. R. Wallach, *Symmetry, Representations and Invariants*, Graduate Texts in Mathematics 255, Springer Dordrecht Heidelberg London New-York, 1985.
- [12] A. Guichardet, *Cohomologie des groupes topologiques et des algèbre de Lie*, Cedec, Paris, 1980.
- [13] R. Hermann, Analytic Continuation of Group Representations IV, *Comm. Math. Phys.* 5 (1967), 131-156.
- [14] L. Hörmander, *The analysis of linear partial differential operators*, Vol. 3, Section 18.5, Springer-Verlag, Berlin, Heidelberg, New-York, 1985.
- [15] A. Joseph, Minimal Realizations and Spectrum Generating Algebras, *Comm. Math. Phys.* 36 (1974), 325-338.
- [16] A. Joseph, The minimal orbit in a simple Lie algebra and its associated maximal ideal, *Ann. Sci. Ecole Norm. Sup.* 9 (1976), 1-30.
- [17] D. Kazhdan, B. Pioline, A. Waldron, Minimal representations, spherical vectors and exceptional theta series, *Comm. Math. Phys.* 226 (2002), 140.
- [18] M. Levy-Nahas, Deformation and Contraction of lie algebras, *J. Math. Phys.* 8, 6 (1967), 1211-1222.
- [19] M. Levy-Nahas, First Order deformations of Lie algebras Representations, $E(3)$ and Poicaré Examples, *Comm. Math. Phys.* 9 (1968), 242-266.
- [20] M. Lesimple and G. Pinczon, Deformations of Lie group and Lie algebra representations, *J. Math. Phys.* 34, 9 (1993), 4251-4272.
- [21] A. Nijenhuis and R. W. Richardson, Cohomology and deformations in graded Lie algebras, *Bull. Amer. Math. Soc.* 72 (1966), 1-29.
- [22] A. Nijenhuis and R. W. Richardson, Deformations of Lie Algebras Structures, *J. Math. Mech.* 17 (1967), 89-105.
- [23] A. Nijenhuis and R. W. Richardson, Deformations of homomorphisms of Lie groups and Lie Algebras, *Bull. Amer. Math. Soc.* 73 (1967), 175-179.
- [24] A. Voros, An Algebra of Pseudo differential operators and the Asymptotics of Quantum Mechanics, *J. Funct. Anal.* 29 (1978), 104-132.

UNIVERSITÉ DE LORRAINE, SITE DE METZ, UFR-MIM, DÉPARTEMENT DE MATHÉMATIQUES,
BÂTIMENT A, ILE DU SAULCY, CS 50128, F-57045, METZ CEDEX 01, FRANCE.

E-mail address: `benjamin.cahen@univ-lorraine.fr`